J. Fluid Mech. (1987), *vol.* 175, *pp.* 143–155
Printed in Great Britain

Second-order low-frequency wave forces on a vertical circular cylinder

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(Received 8 April 1986 and in revised form 16 June 1986)

Using the multiple-scale perturbation method, the diffraction of a nonlinear nearly periodic wavetrain by a vertical circular cylinder is investigated. The envelope of the incident wavetrain is assumed to modulate slowly in the direction of wave propagation. The relationship between the envelopes of incident and scattered waves is derived. It is shown that second-order scattered set-down waves propagate only at the long-wave velocity $(gh)^{\frac{1}{2}}$. The formula for low-frequency wave forces acting on the cylinder is presented. The low-frequency wave forces, which are second-order quantities, are caused by set-down waves beneath the wavetrain and the results of the self-interactions of the leading-order first harmonic wave components. Numerical solutions are presented for the case where the wave envelope varies sinusoidally.

1. Introduction

Wave forces acting on piles have been the subject of investigation by many researchers for many years. The linear diffraction theory for small-amplitude Stokes waves incident on a vertical circular cylinder was due to Havelock (1940) for deep water and MacCamy & Fuchs (1954) for a finite depth. Several attempts have been made by various researchers to include the effects of finite amplitude (Chakrabarti 1972; Raman, Jothishanker & Venkatanarasaiah 1977). However, as pointed out by Isaacson (1977), these nonlinear theories failed to satisfy all the hydrodynamic boundary conditions. Using the method of Fourier–Bessel integral representation for the second-order velocity potential, Hunt & Baddour (1981) were able to correct the errors and found the second-order wave forces on a circular cylinder in deep water. Hunt & Williams (1982) extended the theory for general water depths.

In Hunt and his colleagues' work, the second-order Stokes wave was assumed to be uniform. Therefore, their second-order wave forces have a frequency which is twice that of the carrier-wave frequency. In practical engineering design, however, it is more important to find the low-frequency second-order wave forces since the natural frequency of the structure and mooring system is usually much lower than the carrier frequency. Therefore, it is the objective of this paper to find the second-order low-frequency wave forces on a circular cylinder by a second-order Stokes wave whose envelope modulates slowly in both space ant time.

The multiple scales perturbation method is employed here to study the three-dimensional problem. The same technique has been developed by Agnon & Mei (1985) for a two-dimensional problem. For the present problem, the relationship between the incident-wave envelope and that of the scattered-wave envelope is first found.

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The scattered-wave envelope propagates in the radial direction with the group velocity of the incident-wave envelope. The second-order scattered set-down waves are shown to be propagating only with the long-wave speed, $(gh)^{\frac{1}{2}}$. Unlike the two-dimensional problem studied by Agnon & Mei, the self-interaction of the first-order scattered waves does not generate any forcing terms for the scattered set-down waves.

The formula for the low-frequency wave forces acting on the cylinder is derived. The second-order, low-frequency wave forces are caused by the set-down waves as well as the self-interactions of the leading-order first harmonic wave components. Numerical results are presented for the case where the envelope of the incident wavetrain is a sine function.

The theory presented herein is restricted to the flow regime wherein flow separation does not occur. Therefore the wave amplitude must be much smaller than the diameter of the cylinder.

2. Formulation of the problem

2.1. *Governing equations*

Consider a vertical circular cylinder with radius a, being fixed on a horizontal sea bottom of depth h. Assuming that the fluid is inviscid and the fluid is irrotational, the velocity potential, $\Phi(x,y,z,t)$, satisfies the Laplace equation

$$\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0 \quad (-h<z<\zeta), \tag{2.1}$$

where ζ denotes the free-surface elevation. The subscripts represent partial differentiations. The Cartesian coordinate system (x,y,z) is fixed on the undisturbed free surface; the z-axis coincides with the vertical axis of the cylinder and points upward. On the sea bottom, $z=-h$, and the cylinder, $r=a$, the normal flux vanishes. Thus

$$\Phi_z=0 \quad (z=-h), \tag{2.2}$$

$$\Phi_r=0 \quad (r=a). \tag{2.3}$$

A cylindrical polar coordinate system (r,θ,z) has also been employed herein for convenience. On the free surface, the kinematic condition requires

$$\zeta_t+\zeta_x\Phi_x+\zeta_y\Phi_y=\Phi_z \quad (z=\zeta), \tag{2.4}$$

while the dynamic condition requires

$$g\zeta+\Phi_t+\tfrac{1}{2}(\Phi_x^2+\Phi_y^2+\Phi_z^2)=0 \quad (z=\zeta). \tag{2.5}$$

2.2. *Perturbation equations*

We assume that the incident wavetrain, propagating in the positive x-direction, is nearly periodic with frequency ω. The envelope of the incident wavetrain modulates slowly in both x-direction and time t. For small-amplitude waves $kA=O(\epsilon)\ll 1$, where k is the wavenumber of the carrier waves, and the length and time scales of the envelope modulation are $O(\epsilon^{-1})$ times $2\pi/k$ and $2\pi/\omega$, respectively. Since similar contrast in scale is expected in the scattered-wave field, we introduce the following expansions in terms of the fast (x, y, z, t) and the slow $(x_1 = \epsilon x,\, y_1 = \epsilon y,\, t_1 = \epsilon t)$ variables:

$$\left.\begin{aligned}\Phi &= \epsilon\Phi_1(x,y,z,t,x_1,y_1,t_1)+\epsilon^2\Phi_2(x,y,z,t,x_1,y_1,t_1)+\dots,\\ \zeta &= \epsilon\zeta_1(x,y,t,x_1,y_1,t_1)+\epsilon^2\zeta_2(x,y,t,x_1,y_1,t_1)+\dots.\end{aligned}\right\} \tag{2.6}$$

Substituting (2.6) into the governing equations (2.1)–(2.5) and collecting the terms in the same order of magnitude, we obtain, from the Laplace equation in $-h < z < 0$:

$$\Phi_{1xx}+\Phi_{1yy}+\Phi_{1zz} = 0, \tag{2.7a}$$

$$\Phi_{2xx}+\Phi_{2yy}+\Phi_{2zz} = -2\Phi_{1xx_1}-2\Phi_{1yy_1}, \tag{2.7b}$$

$$\dots,$$

from the free-surface conditions on $z = 0$

$$\Phi_{1tt}+g\Phi_{1z} = 0, \tag{2.8a}$$

$$\Phi_{2tt}+g\Phi_{2z} = -2\Phi_{1tt_1}-\left[\tfrac{1}{2}(\Phi_{1x}^2+\Phi_{1y}^2+\Phi_{1z}^2)-\frac{1}{g}\,\Phi_{1t}\,\Phi_{1zt}\right]_t-(\Phi_{1x}\,\Phi_{1t})_x-(\Phi_{1y}\,\Phi_{1t})_y,$$

$$\dots, \tag{2.8b}$$

$$\zeta_1 = -\frac{1}{g}\,\Phi_{1t}, \tag{2.9a}$$

$$\zeta_2 = -\frac{1}{g}\,[\Phi_{2t}+\Phi_{1tz}\,\zeta_1+\Phi_{1t_1}+\tfrac{1}{2}(\Phi_{1x}^2+\Phi_{1y}^2+\Phi_{1z}^2)], \tag{2.9b}$$

from the no-flux condition on the sea bottom $(z = -h)$

$$\Phi_{1z} = \Phi_{2z} = \dots = 0 \quad (z = -h), \tag{2.10}$$

and on the cylinder $(r = a)$

$$\Phi_{1r} = 0, \tag{2.11a}$$

$$\Phi_{2r} = -\Phi_{1r_1}, \tag{2.11b}$$

$$\dots,$$

where $r_1 = (x_1^2+y_1^2)^{\frac{1}{2}}$, and $r_1 = \epsilon r$.

The solution is sought in terms of harmonics with respect to the fast time variables, i.e.

$$\Phi_n = \sum_{m=-n}^{n} \Phi_{nm}\,\mathrm{e}^{-\mathrm{i}m\omega t}, \tag{2.12a}$$

$$\zeta_n = \sum_{m=-n}^{n} \zeta_{nm}\,\mathrm{e}^{-\mathrm{i}m\omega t}, \tag{2.12b}$$

with $\zeta_{nm} = \zeta^*_{n-m}$, etc., where the '*' denotes the complex conjugate. We note that Φ_{n0} and ζ_{n0} denote the wave fields which vary slowly in time.

2.3. *Low-frequency forces on the cylinder*

The pressure at any point in the fluid is given as

$$p = \rho gz-\epsilon\rho\Phi_{1t}+\epsilon^2\rho[\Phi_{2t}+\tfrac{1}{2}(|\Phi_{1x}|^2+|\Phi_{1y}|^2+\Phi_{1z}|^2)]+O(\epsilon^3). \tag{2.13}$$

The total horizontal wave force on the cylinder in the x-direction can be obtained, in principle, by integrating the x-component of the pressure over the surface of the cylinder, i.e.

$$F_x = \int_0^{2\pi}\int_{-h}^{\zeta} a(-p)\cos\theta\,\mathrm{d}z\,\mathrm{d}\theta \quad (r = a). \tag{2.14}$$

Since we are interested in the low-frequency forces, which depend on only the slow time variable, (2.14) can be reduced to be

$$\overline{F}_x = \rho a \int_{-h}^{0}\int_{0}^{2\pi} \Phi_{10t_1} \cos\theta \, \mathrm{d}\theta \, \mathrm{d}z + \rho a \int_{0}^{2\pi} \cos\theta \left\{ [(-\mathrm{i}\omega \Phi_{11} \zeta_{11}^{*} + \mathrm{c.c.})|_{z=0} + g|\zeta_{11}|^2] + \int_{-h}^{0} (|\Phi_{11x}|^2 + |\Phi_{11y}|^2 + |\Phi_{11z}|^2) \, \mathrm{d}z \right\} \mathrm{d}\theta \quad (r=a), \quad (2.15)$$

where the first term denotes the low-frequency wave forces induced by the set-down waves beneath the wavetrain, while the second term is the result of the self-interactions of the leading-order first harmonic wave component Φ_{11}. In the following sections, analysis is presented to find the solutions for Φ_{11} and Φ_{10} so as to calculate the flow-frequency forces from (2.15).

3. The first-order first harmonic potential, Φ_{11}

The first-order first harmonic potential, Φ_{11}, satisfies the following equations:

$$\Phi_{11xx} + \Phi_{11yy} + \Phi_{11zz} = 0 \quad (-h < z < 0), \quad (3.1)$$

$$\Phi_{11z} - \sigma \Phi_{11} = 0, \quad \sigma = \omega^2/g \quad (z = 0), \quad (3.2)$$

$$\Phi_{11z} = 0 \quad (z = -h), \quad (3.3)$$

$$\Phi_{11r} = 0 \quad (r = a). \quad (3.4)$$

In addition, the radiation condition requires that the disturbance caused by the circular cylinder must be outgoing at infinity.

If the leading-order incident wave potential is given as

$$\Phi_{11}^{\mathrm{I}} = A \frac{\cosh k(z+h)}{\sinh kh} \mathrm{e}^{\mathrm{i}kx} = \sum_{n=0}^{\infty} A \frac{\cosh k(z+h)}{\sinh kh} \beta_n J_n(kr) \cos n\theta, \quad (3.5a)$$

with

$$\sigma = k \tanh kh, \quad (3.5b)$$

where $\beta_n = 1$ for $n = 0$ and $\beta_n = 2\mathrm{i}^n$ for $n \geqslant 1$, the solution to the problem given in (3.1)–(3.4) has been given by MacCamy & Fuchs (1954) and can be expressed as

$$\Phi_{11} = \frac{\cosh k(z+h)}{\sinh kh} \sum_{n=0}^{\infty} \beta_n \left\{ A J_n(kr) - B \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right\} \cos n\theta, \quad (3.6)$$

where H_n is the Hankel function of the first kind. In (3.6) the prime denotes the derivative of the function with respect to the argument. $A(x_1, t_1)$ is the prescribed incident-wave envelope and $B(x_1, y_1, t_1)$ is the envelope of scattered waves to be determined. Through the boundary condition (3.4) A and B are related:

$$A(x_1, t_1) = B(x_1, y_1, t_1) \quad \text{on } r = a. \quad (3.7)$$

To obtain the slow modulation of the scattered-wave envelope, B, we must examine the second-order first harmonic problem. The governing equations for Φ_{21} are

$$\Phi_{21xx} + \Phi_{21yy} + \Phi_{21zz} = \frac{\cosh k(z+h)}{\sinh kh} \left\{ -2\mathrm{i}kA_{x_1} \mathrm{e}^{\mathrm{i}kx} + 2 \sum_{n=0}^{\infty} (B_{x_1} G_{nx} + B_{y_1} G_{ny}) \right\} \quad (-h < z < 0), \quad (3.8a)$$

$$\Phi_{21z}-\sigma\Phi_{21} = \frac{2\mathrm{i}\omega}{\sigma_1}\left\{A_{t_1}\,\mathrm{e}^{\mathrm{i}kx}-\sum_{n=0}^{\infty} B_{t_1} G_n\right\} \quad (z=0), \tag{3.8b}$$

$$\Phi_{21z} = 0 \quad (z=-h), \tag{3.8c}$$

$$\Phi_{21r} = -\Phi_{11r_1} \quad (r=a), \tag{3.8d}$$

where
$$\sigma_1 = \tanh kh, \quad G_n = \beta_n \frac{J_n'(ka)}{H_n'(ka)} H_n(kr)\cos n\theta. \tag{3.8e}$$

The solutions for Φ_{21} can be split into two parts: the incident wave component Φ^I_{21} and the scattered-wave component Φ^S_{21}. Thus

$$\Phi_{21} = \Phi^S_{21}+\Phi^I_{21}. \tag{3.9}$$

The incident-wave component Φ^I_{21} can be readily expressed as (e.g. Mei 1983)

$$\Phi^I_{21} = -\frac{\mathrm{i}(z+h)\sinh k(z+h)}{\sinh kh} A_{x_1}\,\mathrm{e}^{\mathrm{i}kx}. \tag{3.10}$$

Furthermore, $A(x_1, t_1)$ satisfies the conservation equation

$$A_{t_1}+C_g A_{x_1} = 0, \tag{3.11}$$

where C_g is the group velocity $\mathrm{d}\omega/\mathrm{d}k$, which implies

$$A = A(x_1-C_g t_1). \tag{3.12}$$

We propose the solution for Φ^S_{21} in the following form:

$$\Phi^S_{21} = \frac{1}{k}\frac{(z+h)\sinh k(z+h)}{\sinh kh}\sum_{n=0}^{\infty}(B_{x_1}G_{nx}+B_{y_1}G_{ny})+\Psi^S_{21}. \tag{3.13}$$

Substituting (3.9)–(3.13) into (3.8), we obtain a set of governing equations for Ψ^S_{21}:

$$\Psi^S_{21xx}+\Psi^S_{21yy}+\Psi^S_{21zz} = 0 \quad (-h<z<0), \tag{3.14a}$$

$$\Psi^S_{21z}-\sigma\Psi^S_{21} = -\frac{2\mathrm{i}\omega}{\sigma_1}\sum_{n=0}^{\infty}\left\{B_{t_1}G_n-\frac{\mathrm{i}}{kg}C_g(B_{x_1}G_{nx}+B_{y_1}G_{ny})\right\} \equiv P \quad (z=0), \tag{3.14b}$$

$$\Psi^S_{21z} = 0 \quad (z=-h), \tag{3.14c}$$

$$\Psi^S_{21r} = -\Phi_{11r_1}-\Phi^I_{21r} - \frac{1}{k}\frac{(z+h)\sinh k(z+h)}{\sinh kh}\sum_{n=0}^{\infty}(B_{x_1}G_{nx}+B_{y_1}G_{ny})_r \quad (r=a). \tag{3.14d}$$

To ensure the uniqueness of the solution to (3.14), a radiation boundary condition is imposed on the scattered-wave potential, i.e.

$$r^{\frac{1}{2}}\left(\frac{\partial}{\partial r}-\mathrm{i}k\right)\Psi^S_{21}\to 0 \quad \text{as } r\to\infty. \tag{3.15}$$

Note that for later use the right-hand side of (3.14*b*) has been denoted as P.

Since the boundary-value problem for Ψ^S_{21} is a linear one, we can construct the solution as the addition of two solutions: one responds to the inhomogeneous free-surface boundary condition and the other one results from the inhomogeneous condition on the cylinder. Thus

$$\Psi^S_{21} = \tilde{\Psi}^S_{21}+\hat{\Psi}^S_{21}, \tag{3.16}$$

where $\tilde{\Psi}^{\mathrm{S}}_{21}$ is the solution of the following equations:

$$\tilde{\Psi}^{\mathrm{S}}_{21xx}+\tilde{\Psi}^{\mathrm{S}}_{21yy}+\tilde{\Psi}^{\mathrm{S}}_{21zz}=0 \quad (-h<z<0), \tag{3.17a}$$

$$\tilde{\Psi}^{\mathrm{S}}_{21z}-\sigma\tilde{\Psi}^{\mathrm{S}}_{21}=P \quad (z=0), \tag{3.17b}$$

$$\tilde{\Psi}^{\mathrm{S}}_{21z}=0 \quad (z=-h), \tag{3.17c}$$

$$\tilde{\Psi}^{\mathrm{S}}_{21r}=0 \quad (r=a), \tag{3.17d}$$

$$r^{\frac{1}{2}}\left(\frac{\partial}{\partial r}-\mathrm{i}k\right)\tilde{\Psi}^{\mathrm{S}}_{21}\rightarrow 0 \quad \text{as } r\rightarrow\infty, \tag{3.17e}$$

and $\hat{\Psi}^{\mathrm{S}}_{21}$ satisfies the following governing equations

$$\hat{\Psi}^{\mathrm{S}}_{21xx}+\hat{\Psi}^{\mathrm{S}}_{21yy}+\hat{\Psi}^{\mathrm{S}}_{21zz}=0 \quad (-h<z<0), \tag{3.18a}$$

$$\hat{\Psi}^{\mathrm{S}}_{21z}-\sigma\hat{\Psi}^{\mathrm{S}}_{21}=0 \quad (z=0), \tag{3.18b}$$

$$\hat{\Psi}^{\mathrm{S}}_{21z}=0 \quad (z=-h), \tag{3.18c}$$

$$\hat{\Psi}^{\mathrm{S}}_{21r}=-\Phi_{11r_1}-\Phi^{\mathrm{I}}_{21r} -\frac{1}{k}\frac{(z+h)\sinh k(z+h)}{\sinh kh}\sum_{n=0}^{\infty}(B_{x_1}G_{nx}+B_{y_1}G_{ny})_r \quad (r=a), \tag{3.18d}$$

$$r^{\frac{1}{2}}\left(\frac{\partial}{\partial r}-\mathrm{i}k\right)\hat{\Psi}^{\mathrm{S}}_{21}\rightarrow 0 \quad \text{as } r\rightarrow\infty. \tag{3.18e}$$

The right-hand side of (3.17*b*) takes the following asymptotic form as $r\rightarrow\infty$:

$$P\sim-(B_{t_1}+C_g B_{r_1})\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr}\sum_{n=0}^{\infty}\frac{2\mathrm{i}\omega}{\sigma_1}g_n\cos n\theta+O(r^{-\frac{3}{2}}), \tag{3.19}$$

where
$$g_n=\beta_n\frac{J_n'(ka)}{H_n'(ka)}\mathrm{e}^{-\frac{1}{4}\mathrm{i}(2n+1)\pi}. \tag{3.20}$$

Since the leading-order term given in (3.19) is a part of solutions for the homogeneous problem of (3.17), the particular solution for (3.17) must behave as rP or $r^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}kr}$ as $r\rightarrow\infty$, which violates the radiation boundary condition (3.17*e*), and causes the inconsistency in the asymptotic expansion, i.e. $|\Phi_{21}|\gg|\Phi_{11}|$ as $r\rightarrow\infty$. To eliminate this difficulty, we require

$$B_{t_1}+C_g B_{r_1}=0, \quad \text{or} \quad B=B(r_1-C_g t_1,\theta). \tag{3.21}$$

Therefore, the scattered-wave envelope travels in the radial direction with a speed C_g. The free-surface boundary condition for $\tilde{\Psi}^{\mathrm{S}}_{21}$ becomes

$$\tilde{\Psi}^{\mathrm{S}}_{21z}-\sigma\tilde{\Psi}^{\mathrm{S}}_{21}=Q \quad (z=0), \tag{3.22a}$$

where

$$Q=-\sum_{n=0}^{\infty}\frac{2\mathrm{i}\omega}{\sigma_1}\left\{B_{t_1}\left[G_n-g_n\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr}\cos n\theta\right]\right.$$
$$\left.-\frac{\mathrm{i}}{k}C_g\left[B_{x_1}G_{nx}+B_{y_1}G_{ny}-B_{r_1}\mathrm{i}k\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr}g_n\cos n\theta\right]\right\}=-\sum_{n=0}^{\infty}\frac{2\omega\mathrm{i}}{\sigma_1}F_n\cos n\theta. \tag{3.22b}$$

The solutions for the second-order potentials Ψ_{21} can be obtained by Weber's method (Hunt & Williams 1982). They do not play any role in calculating the low-frequency forces and will not be presented here.

Combining (3.21) with (3.7) and (3.12), we obtain

$$B(r_1 - C_g t_1, \theta) = A[r_1 - C_g t_1 - \epsilon a(1 - \cos\theta)]. \tag{3.23}$$

Since the envelope of the incident wavetrain, $A(x_1 - C_g t_1)$, is prescribed, the envelope of the scattered waves can be found from (3.23) by substituting x_1 by $[r_1 - \epsilon a(1-\cos\theta)]$. The solution for Φ_{11} is, therefore, completed. We now turn our attention to the solution for Φ_{10}.

4. The first-order zeroth harmonic potential

From the zeroth harmonic of (2.7*a*)–(2.11*a*), the short-scale variation of Φ_{10} is governed by the following equations:

$$\Phi_{10xx} + \Phi_{10yy} + \Phi_{10zz} = 0 \quad (-h < z < 0), \tag{4.1a}$$

$$\Phi_{10z} = 0 \quad (z = 0 \text{ and } -h), \tag{4.1b}$$

$$\Phi_{10r} = 0 \quad (r = a). \tag{4.1c}$$

The only possible solution for (4.1) describes a circulation motion; i.e. $\Phi_{10} \sim \theta = \tan^{-1}(x/y)$. This type of solution is, however, discarded in the present analysis since the incident waves do not contain the first-order low-frequency velocity. Therefore, $\Phi_{10} = \Phi_{10}(x_1, y_1, t_1)$ is independent of fast variables.

The long-scale equations for Φ_{10} could be obtained from the third-order zeroth harmonic equations of the basic perturbation equations. Alternatively, following Agnon & Mei's (1985) approach, we substitute the solution series (2.6) into the continuity equation

$$\zeta_t + \left[\int_{-h}^{\zeta} \Phi_x \,\mathrm{d}z\right]_x + \left[\int_{-h}^{\zeta} \Phi_y \,\mathrm{d}z\right]_y = 0, \tag{4.2}$$

and collect the third-order terms. Thus

$$\begin{aligned}
&\zeta_{3t} + \zeta_{2t_1} + \zeta_{1t_2} + \left[\int_{-h}^{0} \Phi_{1x}\,\mathrm{d}z\right]_{x_2} + \left[\int_{-h}^{0} \Phi_{1y}\,\mathrm{d}z\right]_{y_2} \\
&\quad + \left[\int_{-h}^{0} (\Phi_{1x_1} + \Phi_{2x})\,\mathrm{d}z + \zeta_1 \Phi_{1x}\right]_{x_1} + \left[\int_{-h}^{0} (\Phi_{1y_1} + \Phi_{2y})\,\mathrm{d}z + \zeta_1 \Phi_{1y}\right]_{y_1} \\
&\quad + \left[\int_{-h}^{0} (\Phi_{1x_2} + \Phi_{2x_1} + \Phi_{3x})\,\mathrm{d}z\right]_{x} + \left[\int_{-h}^{0} (\Phi_{1y_2} + \Phi_{2y_1} + \Phi_{3y})\,\mathrm{d}z\right]_{y} \\
&\quad + (\zeta_1 \Phi_{1x_1} + \zeta_1 \Phi_{2x} + \zeta_2 \Phi_{1x} + \tfrac{1}{2}\zeta_1^2 \Phi_{1xz})_x + (\zeta_1 \Phi_{1y_1} + \zeta_1 \Phi_{2y} + \zeta_2 \Phi_{1y} + \tfrac{1}{2}\zeta_1^2 \Phi_{1yz})_y = 0.
\end{aligned} \tag{4.3}$$

Substituting (2.12) into (4.3) and collecting the zeroth harmonic terms, we obtain

$$\begin{aligned}
&h\left\{\Phi_{10x_1x_1} + \Phi_{10y_1y_1} - \frac{1}{gh}\Phi_{10t_1t_1} + \frac{2\omega k}{\sigma_1^2 gh}(|A|^2)_{x_1} + \frac{k^2}{gh}(|A|^2)_{t_1} - \frac{k^2}{\sigma_1^2 gh}(|A|^2)_{t_1}\right\} \\
&\qquad\qquad + h(\Phi_{30xx} + \Phi_{30yy}) = F(x, y, z, t),
\end{aligned} \tag{4.4}$$

where $F(x,y,z,t)$ is a function of fast variables and is the result of cross-products between incident waves and scattered waves as well as the self-products of scattered waves. We remark here that because of the three-dimensionality the scattered-wave amplitude must be a function of r, the fast variable. In fact, from (3.6) or the required radiation boundary condition the scattered-wave field behaves as $r^{-\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}(kr-\omega t)}$ as kr

becomes large. Therefore the self-product of the scattered-wave field must be a function of vast variables. The function F contributes only to the third-order solution of Φ_{30}. Equating all terms involving only the slow variables, we obtain the boundary-value problem for Φ_{10}:

$$\Phi_{10x_1x_1}+\Phi_{10y_1y_1}-\frac{1}{gh}\Phi_{10t_1t_1}=-\frac{k^2}{\sigma_1^2 gh}(|A|^2)_{x_1}\left[\frac{2\omega}{k}-C_g(\sigma_1^2-1)\right], \tag{4.5}$$

$$\Phi_{10r_1}=0 \quad (r=a), \tag{4.6}$$

where (3.12) has been employed.

The right-hand side of (4.5) represents the forcing terms for the set-down waves in the incident waves. We can, therefore, separate the total potential into two parts:

$$\Phi_{10}=\Phi_{10}^{\mathrm{I}}+\Phi_{10}^{\mathrm{S}}, \tag{4.7}$$

where Φ_{10}^{I} is the incident-wave potential, which is given as

$$\Phi_{10x_1}^{\mathrm{I}}=\frac{1}{C_g^2-gh}\frac{k^2}{\sigma_1^2}\left[\frac{2\omega}{k}-C_g(\sigma_1^2-1)\right]|A|^2, \tag{4.8}$$

and Φ_{10}^{S} denotes the scattered-wave potential, satisfying

$$\Phi_{10x_1x_1}^{\mathrm{S}}+\Phi_{10y_1y_1}^{\mathrm{S}}-\frac{1}{gh}\Phi_{10t_1t_1}^{\mathrm{S}}=0, \tag{4.9}$$

$$\Phi_{10r_1}^{\mathrm{S}}=-\Phi_{10r_1}^{\mathrm{I}} \quad (r=a). \tag{4.10}$$

The scattered low-frequency waves propagate with the shallow water wave velocity, $(gh)^{\frac{1}{2}}$. To find the specific solution form for the scattered-wave potential, we must describe the envelope of the incident waves. However, without losing generality, we assume that Φ_{10}^{I} can be written in a form of Fourier series

$$\begin{aligned}\Phi_{10}^{\mathrm{I}}&=C_0x_1-\frac{\mathrm{i}}{k_0}\sum_{n=1}^{\infty}C_n\,\mathrm{e}^{\mathrm{i}n(k_0x_1-\omega_0t_1)}+\text{c.c.}\\&=C_0x_1-\frac{\mathrm{i}}{k_0}\sum_{n=1}^{\infty}C_n\,\mathrm{e}^{-\mathrm{i}n\omega_0t_1}\sum_{m=0}^{\infty}\beta_mJ_m(k_0nr_1)\cos n\theta+\text{c.c.},\end{aligned} \tag{4.11}$$

where k_0 denotes the wavenumber of the envelope of incident waves and $\omega_0=C_gk_0$. The coefficients C_n $(n=0,1,2,\ldots)$ are determined from (4.8) for a specific A. The solution to (4.9) and (4.10) can be obtained as

$$\begin{aligned}\Phi_{10}^{\mathrm{S}}=\frac{C_0a^2}{r_1}\cos\theta+\frac{\mathrm{i}}{k_0}\sum_{n=1}^{\infty}C_n\,\mathrm{e}^{\mathrm{i}n\omega_0t_1}\sum_{m=0}^{\infty}\beta_m\frac{J_m'(k_0na)}{\alpha_0H_m'(\alpha_0k_0na)}&\\\times H_m(\alpha_0k_0nr_1)\cos m\theta+\text{c.c.}&\end{aligned} \tag{4.12}$$

Thus the long-scale potential can be found from (4.7), (4.11) and (4.12). We remark here that the scattered-wave potential satisfies the radiation boundary condition:

$$r_1^{\frac{1}{2}}\left(\frac{\partial}{\partial r_1}-\mathrm{i}k_n\right)\Phi_{10}^{\mathrm{S},n}\to 0 \quad \text{as } r_1\to\infty, \tag{4.13}$$

where $k_n=\alpha_0k_0n$ and $\Phi_{10}^{\mathrm{S},n}$ is the nth wave component of the scattered-wave potential Φ_{10}^{S}. The complete solution for the slow variable potential Φ_{10} is the summation of (4.11) and (4.12) as suggested by (4.7).

5. Low-frequency forces on the cylinder

Using the solutions for Φ_{11}, (3.6), and Φ_{10}, (4.11) and (4.12), we can now rewrite the formula for low-frequency wave forces on the cylinder in the following form:

$$\overline{F}_x = f_1 + f_2 \quad (r = a), \tag{5.1a}$$

where

$$\begin{aligned} f_1 &= \rho a h \int_0^{2\pi} \Phi_{10t_1} \cos\theta \, \mathrm{d}\theta \\ &= -2\rho a h \omega_0 \pi \frac{\mathrm{i}}{k_0} \sum_{n=1}^{\infty} C_n \, \mathrm{e}^{-\mathrm{i}n\omega_0 t_1} \left[J_1(nk_0 a) - \frac{J_1'(nk_0 a)}{\alpha_0 H_1'(\alpha_0 nk_0 a)} H_1(\alpha_0 nk_0 a) \right] + \text{c.c.} \end{aligned} \tag{5.1b}$$

and

$$\begin{aligned} f_2 &= \rho a \int_0^{2\pi} \cos\theta \left\{ -k\sigma_1 |\Phi_{11}|^2_{z=0} + \int_{-h}^{0} \left(|\Phi_{11r}|^2 + |\Phi_{11z}|^2 + \frac{1}{r^2} |\Phi_{11\theta}|^2 \right) \mathrm{d}z \right\} \mathrm{d}\theta \\ &= -\frac{\rho a \omega^2}{g\sigma_1^2} \int_0^{2\pi} |\phi_{11}|^2 \cos\theta \, \mathrm{d}\theta + \rho a G \int_0^{2\pi} \left[|\phi_{11r}|^2 + k^2 |\phi_{11}|^2 + \frac{1}{a^2} |\phi_{11\theta}|^2 \right] \cos\theta \, \mathrm{d}\theta, \end{aligned} \tag{5.1c}$$

with

$$\phi_{11} = \sigma_1 \Phi_{11}|_{z=0}, \tag{5.1d}$$

$$G = \int_{-h}^{0} \frac{\cosh^2 k(z+h)}{\sinh^2 kh} \, \mathrm{d}z. \tag{5.1e}$$

Note that the cylindrical coordinates (r, θ, z) have been employed in (5.1*c*) for convenience. In principle, (5.1*c*) can be integrated numerically once A is given.

For the cases where the diameter of the cylinder is the same order of magnitude as the wavelength of the carrier waves, we can neglect the $\epsilon a \cos\theta$ term in (3.23) in the neighbourhood of the cylinder. Thus, the boundary condition (3.7) can be approximated as

$$B(r_1, t_1, \theta) = A(-C_g t_1) \quad (r = a). \tag{5.2}$$

Using this approximation, (5.1*c*) can be integrated analytically and gives

$$f_2 = \pi\rho a \left(Gk^2 - \frac{\omega^2}{g\sigma_1^2} \right) \left\{ P_0 P_1^* + \text{c.c.} + \frac{1}{2} \sum_{n=1}^{\infty} (P_n P_{n+1}^* + \text{c.c.}) \right\} + \frac{\rho\pi G}{2a} \sum_{n=1}^{\infty} n(n+1)(P_n P_{n+1}^* + \text{c.c.}), \tag{5.3}$$

where

$$P_n = \beta_n A(-C_g t_1) \left[\frac{2\mathrm{i}}{\pi ka H_n'(ka)} \right], \tag{5.4}$$

and P_n^* is the complex conjugate of P_n.

5.1. *A numerical example*

We consider the incident-wave envelope as sinusoidal; i.e.

$$A = \frac{g}{\omega} A_0 \sin k_0 (x_1 - C_g t_1), \tag{5.5}$$

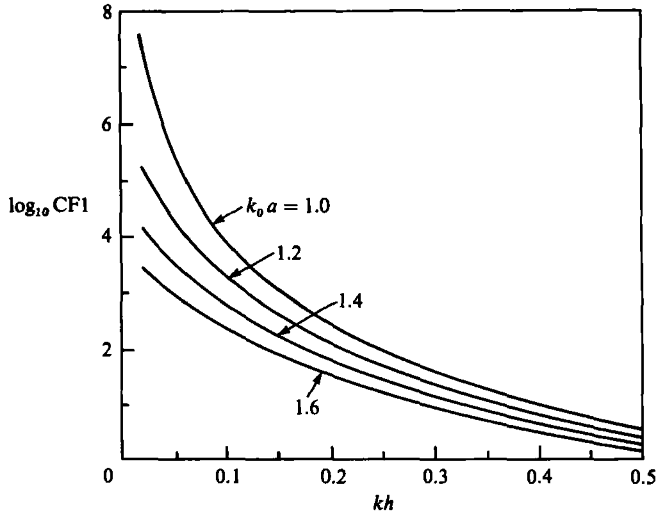

FIGURE 1. Force coefficient CF1 as a function of kh with $O(1)\, k_0 a$.

where A_0 denotes the wave amplitude and $2\pi/k_0$ represents the wavelength of the wave envelope. Substituting (5.5) into (4.10) and (4.11), we obtain

$$\Phi^{\mathrm{I}}_{10} = \frac{1}{2}\left\{\frac{1}{C_g^2-gh}\frac{k^2}{\sigma_1^2}\left[\frac{2\omega}{k}-C_g(\sigma_1^2-1)\right]\frac{g^2A_0^2}{\omega^2}\right\} x_1 - \frac{1}{4k_0}\left(\frac{gA_0}{\omega}\right)^2\left\{\frac{1}{C_g^2-gh}\frac{k^2}{\sigma_1^2}\left[\frac{2\omega}{k}-C_g(\sigma_1^2-1)\right]\right\}\sin 2k_0\,(x_1-C_g t_1). \quad (5.6)$$

It follows, from (5.1*b*), that the low-frequency wave forces induced by the set-down waves can be written as

$$f_1 = \rho h a \omega_0 \pi A_2\left[J_1(2k_0 a)-\frac{J_1'(2k_0 a)}{\alpha_0 H_1'(2k_0\alpha_0 a)}H_1(2\alpha_0 k_0 a)\right]\mathrm{e}^{-2\mathrm{i}\omega_0 t}+\mathrm{c.c.}, \quad (5.7)$$

where A_2 represents the amplitude of the oscillations of the set-down waves. Thus

$$A_2 = \frac{\mathrm{i}}{4k_0}\left(\frac{gA_0}{\omega}\right)^2\left\{\frac{1}{C_g^2-gh}\frac{k^2}{\sigma_1^2}\left[\frac{2\omega}{k}-C_g(\sigma_1^2-1)\right]\right\}. \quad (5.8)$$

We can define the maximum horizontal force coefficient CF1 as follows:

$$\mathrm{CF1} = \max_t \left|\frac{f_1}{\rho g h a k A_0^2}\right|. \quad (5.9)$$

In figure 1 we show the variation of CF1 as a function of both $k_0 a$ and kh with values of $k_0 a$, being $O(1)$. The force coefficient increases rapidly as kh decreases, which is caused by the fact that $C_g \rightarrow gh$, A_2 and $f_1 \rightarrow \infty$ as $kh \rightarrow 0$. The wave forces approach zero when the diameter of the circular cylinder becomes small, i.e. $k_0 a \rightarrow 0$ (figure 2). As shown in figure 2, the force coefficient is very sensitive to the water depth in terms of both magnitude and its dependence on $k_0 a$. For the intermediate water depth, $kh \approx O(1)$, the force coefficients vary oscillatory as a function of $k_0 a$. The amplitudes of oscillations decrease as kh decreases and $k_0 a$ increases. Using the same normalization

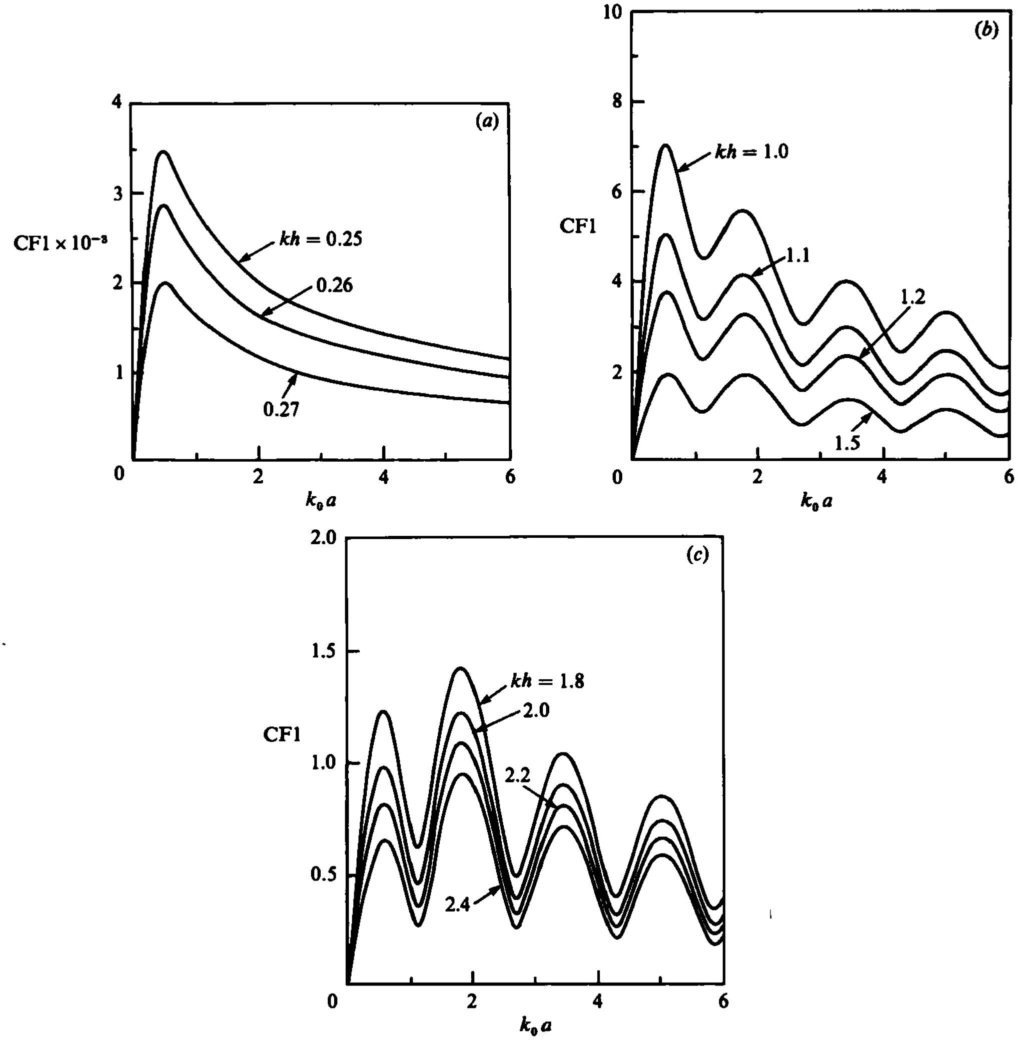

FIGURE 2. Force coefficient CF1 as a function of $k_0 a$: (*a*) $0.25 < kh < 0.27$, (*b*) $1.0 < kh < 1.5$, (*c*) $1.8 < kh < 2.4$.

factor as that defined in (5.9), we can introduce the maximum horizontal force coefficient CF2 corresponding to f_2 as follows:

$$\mathrm{CF2} = \max_t \left| \frac{f_2}{\rho g h a k A_0^2} \right|. \tag{5.10}$$

Equation (5.3) is used in (5.10) to find CF2 as a function of kh and ka. Since f_2 is generated by the self-products of the first-order wave motion, CF2 is not a function of the parameters associated with the envelope, i.e. $k_0 a$. As shown in figure 3, the wave force component has a maximum value at $ka \approx 1.0$ independent of kh. The wave forces also increase as kh decreases.

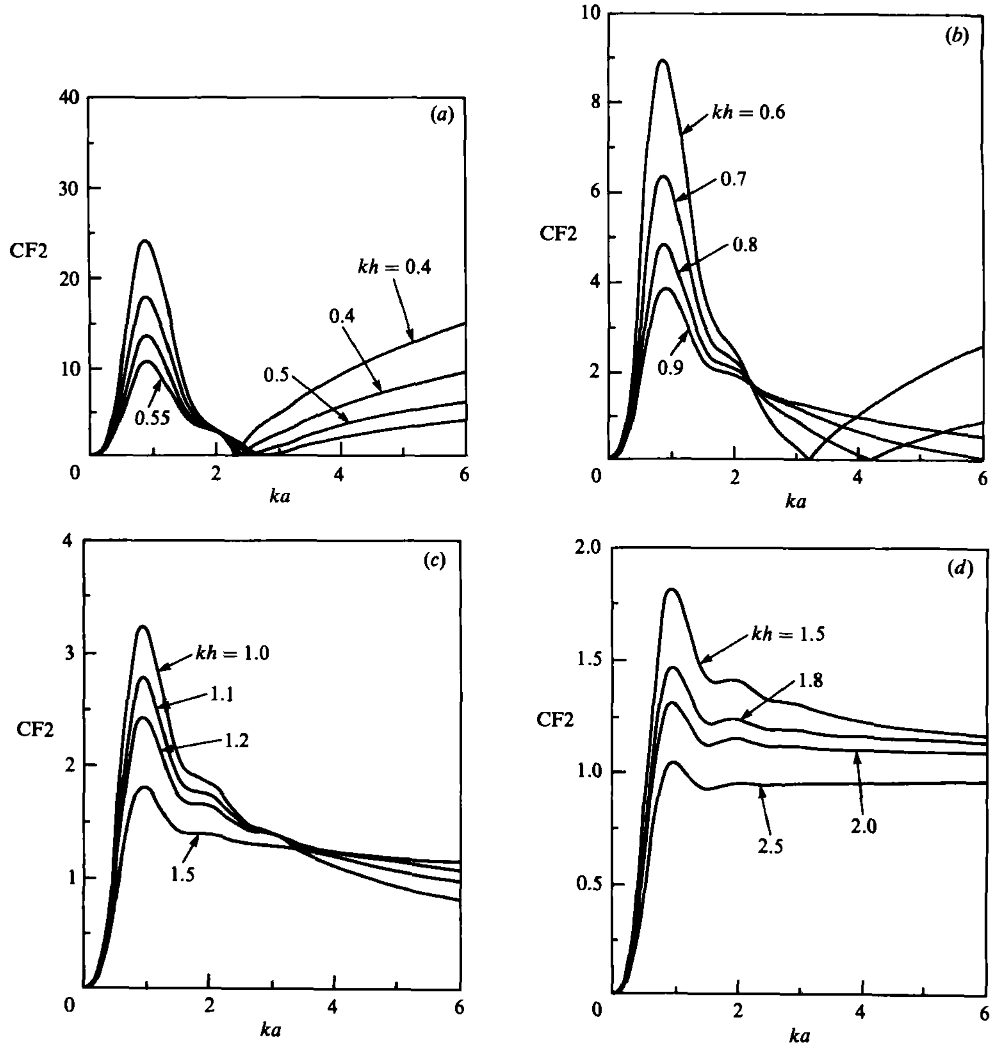

FIGURE 3. Force coefficient CF2 as a function of ka: (a) $0.4 < kh < 0.55$, (b) $0.6 < kh < 0.9$, (c) $1.0 < kh < 1.5$, (d) $1.5 < kh < 2.5$.

6. Concluding remarks

In this paper, we have presented a complete solution for the diffraction of a slowly modulating wavetrain by a vertical circular cylinder, up to the second order of kA. Two important results have been obtained in the process of deriving formula for low-frequency wave forces: (1) the relationship between the incident-wave envelope and the scattered-wave envelope is given in (3.23), which suggests that the scattered-wave envelope propagates in the radial direction with the group velocity C_g of the incident wavetrain, and (2) the scattered second-order set-down waves propagates with the long-wave speed $(gh)^{\frac{1}{2}}$, (4.9). The first result seems to be associated with the geometry of the circular cylinder, but the second result does not. It would be interesting to extend the present theory to problems involving scatterers with arbitrary geometries. Laboratory experiments should also be performed to measure the low-frequency wave forces so as to verify the present theory.

The research work was carried out while C.Z. was visiting Cornell University. The financial supports provided by The Chinese Academy of Sciences, Ministry of Education in China and the School of Civil and Environmental Engineering at Cornell University are appreciated. We acknowledge Mr J.-K. Wu's assistance in obtaining numerical results. The research work was supported, in part, by the New York Sea Grant Institute.

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